the translation of a posthumous memoir of Riemann, most interesting for the extreme importance and vastness of its subject, to which, it is necessary to say, the brevity of the development is little adequate.

As to my translation I ought to mention in the first place that I have tried to be most faithful to the text, for fear that doing otherwise I might alter the true sense of a composition which in many points gives place for doubts of interpretation.

At the end of the translation I have inserted some annotations, made *currenti calamo* and for *internal use* (employing the phrase of the apothecaries), that is to say destined properly for myself alone. Hence such notes either should not be read, or not without the greatest indulgence, because most of them are rather indications destined to serve me as guides to call attention to those points that occasioned me the most difficulty, than comments properly so called and definitive.

There are then very many other points which merit elucidation. You will find in two of my annotations certain allusions to a mode of mine of interpreting the results of the non-Euclidean geometry. If you wish any hint about this you might turn to Professor Cremona, who has had in his hands a manuscript of mine containing the complete *real* construction of the planimetry of Lobachevski. The ground of the researches of Lobachevski lies beyond doubt in the doctrine touched by Riemann. I intend to publish some of my studies on this subject, encouraged by the support which they find in the work of Riemann, only lately come to my cognizance."

"Bologna, 23 July, 1868.

The past year, when no one knew of this fundamental work of Riemann, I had communicated to Cremona a paper of mine in which I gave an interpretation of the non-Euclidean planimetry, which seemed to me satisfactory. Cremona did not judge differently.

At the end I had risked a judgment on the non-Euclidean stereometry, which now I do not think correct. But the article, freed from this slip, will appear in the *Giornale* of Naples, in its original form, save certain additions that I can hazard now, because substantially concordant with some of the ideas of Riemann."

The work referred to is the "Saggio d' interpretazione della geometria non-euclidea" (Giorn. di matem. 6, 1868, pp. 284-312), the first of the articles which, to use the expression of Cremona, "gave to Beltrami as if at a cast that reputation which always widening became universal admiration."

Its genial idea, for which Beltrami had been prepared by his researches on surfaces of constant curvature, an irresistable idea to silence opposers of the non-Euclidean geometry, was to present the example of a surface, regular as the plane and the sphere, in which the lines corresponding to the straights of the plane and to the great circles of the sphere, that is the geodetics, comported themselves as the straights of the non-Euclidean plane. This was a surface of constant negative curvature, a pseudosphere.

But very soon after the publication of this memoir, Helmholtz and Klein expressed grave doubts on certain points of Beltrami's reasoning and mathematics, saying that he had not made certain the existence of surfaces of the type to which he has recourse to represent, without changing its nature, the new system of geometry. It seems that Beltrami tried to dissipate these doubts. At least it seems that to such attempts we owe a memoir where is studied with scrupulous care the surface generated by the rotation of the tractrix about its asymptate with the aim of deducing the elements by a construction simple and exact of the surface itself.

The end aimed at not having been reached, Genocchi repeated the objections in a way still more particularized and energetic, maintaining that it had not been demonstrated that the partial differential equation, characteristic of the surface of constant negative curvature, admits at least one integral satisfying all the conditions imposed on the pseudo-sphere to serve the representation of Beltrami.

There is no trace of response from Beltrami, not even in his correspondence with Genocchi. Perhaps he perceived that unfortunately those objections were well founded.

But the final establishment of that fact, the complete demonstration of the non-existence of regular surfaces of constant negative curvature in all their extent, was not accomplished until after the death of Beltrami, by Hilbert in 1901.

The representation imagined by Beltrami is therefore sufficiently limited, and the beautiful edefice, if it does not fall to the ground, shows itself in solidity and extension less than what was believed. Fortunately the non-Euclidean geometry is now completely vindicated in many simpler ways. Moreover Beltrami's pseudo-sphere had always done harm to the many people who took up the false idea that geodesic geometry on the pseudo-sphere was Bolyai's non-Euclidean geometry, instead of only being an interesting *representation* of it in Euclidean space; just as Bolyai's geometry of goedesics on a limit surface in Lobachevski's space is a representation of Euclidean geometry in Bolyai space.

From this epoch (1868) the interest of Beltrami in non-Euclidean geometry never flagged.

He made an application of the expression given by Lobachevski for the angle of parallelism. In his correspondence with Genocchi he points out the flaw in the argumentation proposed by Carton for demonstrating the postulate of Euclid, presented by Bertrand to the Institute of France December 20, 1869, he discusses the enquiries undertaken to determine who was Schweikart, now so well known as an independent creator of the non-Euclidean geometry, and he gives an appreciation of Hoüel's article in "Sur l'impossibilité de démontrer par une construction plane le postulatum d' Euclide." A public proof of this interest is also the charming communication made to the Accademia dei Lincei March 17, 1889, to present in the proper light the work, whose value and significance were then unknown, of Saccheri, "Euclides ab omni naevo vindicatus."

Beltrami never ceased to meditate on the non-Euclidean geometry even when concentrating all his powers to the study of natural phenomena. A proof of this is his discovery that the general equation of elasticity is bound to the Euclidean postulate. Moreover one of his gifted disciples observes "how he shows, in a certain passage, that he had turned his attention to the way in which physics would be able to profit from hypotheses of a diverse geometric nature of space, a difficult conception, more explicitly advanced by Clifford; nor was he ever able to lose from view those curved spaces, with which he had commenced so triumphantly."

Settled finally at Rome, member of the most celebrated scientific societies of the world, successor to Brioschi as President of the Accademia dei Lincei, Senator of the Realm of Italy, many times chosen by public vote to sit in the Council Superior of Public Instruction, acclaimed master by the entire body of scientists, happy with a devoted wife, yet from 1896 he was undermined by a mysterious malady, and died February 18th, 1900.

*The University of Texas.*

## ON THE PRIMITIVE GROUPS OF CLASS FOUR.

By DR. G. A. MILLER.

The class of a substitution group is the smallest number of elements in any one of its substitutions besides the identity. This definition seems to be due to Camille Jordan,* who made extensive investigations in regard to the primitive groups of small classes. It is well known that the alternating and the symmetric groups are of classes two and three respectively, and that these are the only primitive groups of these two classes. Moreover, Jordan proved that these are the only two classes for which the number of primitive groups is infinite,† and he investigated the problem of determining all the primitive groups whose class does not exceed 13, publishing only a brief outline of his work.‡

In his work on the *Theory of Substitutions*, pages 133 to 138, Netto gives an outline of a proof that there is no primitive group of class four and of degree greater than 8. As this theorem is of great importance in the theory of primitive groups, it appeared desirable to give a more complete proof, based upon some recent theorems. This proof will thus serve as another illustration of the application of these theorems, and it is hoped that it will tend to simplify one step towards the difficult subject of class of primitive substitution groups.

Let $G$ be any primitive group of degree $n>8$ and of class four. The subgroup $(H)$ generated by its substitutions of the form $ab.cd=s_1$, which may be supposed to be contained in $G$, must be invariant, since it includes all the con-

*Jordan, Liouville, vol. 16, 1871, page 383.
†Comptes Rendus, vol. 73, 1871, p. 852.
‡Loc. cit. vol. 75, p. 1757.

jugates of $s_1$. Hence it must be of degree $n$ and transitive. It must be non-abelian since abelian transitive groups are regular. We shall first prove that $H$ must include the regular four-group including $s_1$. This will prove that $G$ is at least doubly transitive and that each one of its substitutions of degree four and order two is contained in a sub-group of order and degree four.

The sub-group $H$ contains at least one substitution $s_2$ which is similar to $s_1$ and non-commutative with it. If $s_1$ and $s_2$ would have only one element in common their commutator $s_1^{-1}s_2^{-1}s_1s_2$ would be of degree three and $G$ could not be of class four.* If they had three common elements they would generate a group of degree 5, which would be transitive since the intransitive groups of this degree are either of class 2 or of class 3. Every transitive group of degree $p$, $p$ being any prime number, must include a cyclic substitution of degree $p$. It has been proved that such a substitution cannot occur in any primitive group whose class exceeds 3 except when the degree is one of the three numbers $p, p+1, p+2$.† Hence $s_1$ and $s_2$ must contain just two common elements; that is *any two non-commutative substitutions of G which are similar to $s_1$ have just two common elements.*

The two substitutions $s_1$ and $s_2$ must therefore generate a dihedral rotation group of degree six whose order is either 6 or 8.‡ In the latter case $s_1$ is evidently included in a regular four-group. In the former case, the group generated by $s_1$ and $s_2$ is the intransitive group obtained by establishing a simple isomorphism between two symmetric groups of degree three. We proceed to prove that in this case $s_1$ must also be included in a regular four-group contained in $H$.

Only three of the substitutions of $H$ which are similar to $s_1$ have been determined, viz: $s_1, s_2, s_1^{-1}s_2s_1$. If all the other similar substitutions were commutative with $s_1$ they would also have to be commutative with each of the other two given conjugates, since these three substitutions are transformed transitively by a sub-group which transforms all the rest among themselves. Hence $H$ includes another substitution $(s_3)$ which is not commutative with $s_1$, and therefore it has just two elements in common with $s_1$.

If the group generated by $s_1, s_2, s_3$ were transitive it would include a regular four-group containing $s_1$ since the degree of this group would be either six or seven and it may be assumed that it would not contain any substitution of order 5. As it would also be positive and of class four there are only two groups which require consideration, viz: $(+abcdef)_{24}$ and $(abcdefg)_{168}$.§ It remains to consider the cases when $s_1, s_2, s_3$ would generate one of the following intransitive groups: (1) A simple isomorphism between two symmetric groups of degree four, 2) A (4, 4) correspondence between these groups, and 3) A (1, 4) correspondence between the symmetric groups of degrees three and four respectively.

In the last two cases $G$ would be at least doubly transitive and hence it

*Bochert, Mathematische Annalen, vol. 40, 1892, p. 159.
†Bulletin of the American Mathematical Society, vol. 4, 1898, p. 141.
‡Loc. cit., vol. 7, 1901, p. 424.
§American Journal of Mathematics, vol. 21, p. 287.

would contain a substitution similar to $s_1$ which would permute its systems; that is, $G$ would contain the regular four-group including $s_1$. In the first case $G$ would contain an additional substitution ($s_4$) similar to $s_1$ and not commutative with $s_1$. If the group generated by $s_1, s_2, s_3, s_4$ were not a simple isomorphism between two symmetric groups of degree five it would clearly include a regular four-group containing a conjugate of $s_1$ and hence also such a group containing $s_1$. As this remark applies to all the following cases and as the substitutions which are similar to $s_1$ generate a transitive group, it follows that *in every primitive group of class four and degree* greater than 8 each substitution of type *ab.cd* is contained in the regular four-group. It may be observed that this applies also to the primitive groups of degrees 7 and 8 but not to those of degrees 5 and 6.

In what follows it may therefore be assumed that $G$ is at least doubly transitive and that each of its substitutions similar to $s_1$ is contained in a regular four-group. It has already been observed that $s_1, s_2$ generate either the positive octic group of degree six or the intransitive group of degree and order six. In either case it may be assumed that $s_1, s_2, s_3$ generate $(+abcdef)_{24}$ since this is the only positive transitive group of degree six and class four which is generated by substitutions similar to $s_1$ and in which each of these substitutions is in a regular four-group. As $G$ must contain some additional substitution ($s_5$) similar to $s_1$ which is not commutative with some one of the three conjugate substitutions of order 2 in $(+abcdef)_{24}$ and as the order of the group generated by $s_1, s_2, s_3, s_5$ must exceed 48, $H$ must include a transitive group of degree seven or the alternating group of degree 6, the latter being the only positive group of degree six which includes $(+abcdef)_{24}$.

The only transitive group of degree seven and of class four is the well known simple group $(abcdefg)_{168}$. It follows from the theorem quoted above that this could not occur in a primitive group whose degree exceeds 9 unless this primitive group were either alternating or symmetric. It could not occur in a primitive group of degree 9 since all its 21 substitutions similar to $s_1$ are conjugate and hence each of these substitutions would be transformed into itself by $5.8=40$ substitutions of the primitive group of degree 9. This is clearly impossible since its order would not be divisible by 5. Hence there is no primitive group of class four and of degree greater than 8.

From what precedes and from the enumeration of the groups of degree 8* it follows that there are just six primitive groups of class four—two of each of the degrees 5 and 6 and one of each of the degrees 7 and 8. Their orders are 10, 20, 60, 120, 168, and 1344 respectively. The first two are the semi-metacyclic and the metacyclic groups† of degree 5. The third and fourth are, respectively, simply isomorphic with the alternating and the symmetric groups of degree six. They are the only instances of transitive groups of degree $n$ and of orders $\frac{1}{2}(n-1)!$ and $(n-1)!$ and have been studied very fully in connection with the theory of equations of degree six. The first of these is known as the

*Cf. American Journal of Mathematics, vol. 21, p. 237.
†Oeuvres de Lagrange, vol, 3, p. 339.

icosahedron rotation group and it is the smallest simple group of composite order. The fifth is very well known in the theory of elliptic modular functions and is the second smallest simple group of composite order. Kirkman remarks:* "Betti, Kronecker, Hermite, and myself have spent much time on this group." The last of these six primitive groups is the holomorph of the group of order 8 which includes no operator of order four.†

*Kirkman, Proceedings of the Manchester Literary and Philosophical Society, vol. 3, p. 65.
†American Journal of Mathematics, loc. cit.

*Leland Stanford University.*

## FACTORS OF A CERTAIN DETERMINANT OF OREER SIX.

By DR. L. E. DICKSON.

The following is an example of the so-called Group-Determinant:*

$$D\equiv\begin{vmatrix} \mathrm{I} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \beta & \mathrm{I} & \alpha & \delta & \varepsilon & \gamma \\ \alpha & \beta & \mathrm{I} & \varepsilon & \gamma & \delta \\ \gamma & \delta & \varepsilon & \mathrm{I} & \alpha & \beta \\ \delta & \varepsilon & \gamma & \beta & \mathrm{I} & \alpha \\ \varepsilon & \gamma & \delta & \alpha & \beta & \mathrm{I} \end{vmatrix}$$

Note that the elements of the first three rows form two cyclic determinants of order three, and that the elements of the last three rows form the same two cyclic determinants. It follows readily that $D$ has the factors $(\mathrm{I}+\alpha+\beta)\pm(\gamma+\delta+\varepsilon)$.

Upon adding to the first column all the remaining columns, we obtain an equal determinant having $\mathrm{I}+\alpha+\beta+\gamma+\delta+\varepsilon$ throughout the first column. Let $D_1$ be the determinant obtained by removing this factor, so that the elements in the first column of $D_1$ are all unity. Subtracting the first row from the remaining rows, we find that

$$D_1=\begin{vmatrix} \mathrm{I}-\alpha & \alpha-\beta & \delta-\gamma & \varepsilon-\delta & \gamma-\varepsilon \\ \beta-\alpha & \mathrm{I}-\beta & \varepsilon-\gamma & \gamma-\delta & \delta-\varepsilon \\ \delta-\alpha & \varepsilon-\beta & \mathrm{I}-\gamma & \alpha-\delta & \beta-\varepsilon \\ \varepsilon-\alpha & \gamma-\beta & \beta-\gamma & \mathrm{I}-\delta & \alpha-\varepsilon \\ \gamma-\alpha & \delta-\beta & \alpha-\gamma & \beta-\delta & \mathrm{I}-\varepsilon \end{vmatrix}$$

From the first row, subtract the third, fourth, and fifth rows, and to the first row add the second row. In the resulting determinant, the elements of the first row are all divisible by $\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon$. Hence

*Its matrix forms the body of a left-hand multiplication-table for the symmetric group on three letters, where

$\mathrm{I}$=identity, $\alpha$=(123), $\beta$=(132), $\gamma$=(12), $\delta$=(13), $\varepsilon$=(23).

Compare Weber, *Algebra*, 2nd Edition, Vol. II, page 124.

$$D_1=(\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon)D_2,$$

where the elements in the first row of $D_2$ are 1, 1, $-1$, $-1$, $-1$; the remaining rows of $D_2$ being identical with the corresponding rows of $D_1$. Next subtract the first column of $D_2$ from the second column, and add the first column to the third, fourth, and fifth columns.

$$\therefore D_2=\begin{vmatrix} \mathrm{I}+\alpha-2\beta & -\alpha+\beta-\gamma+\varepsilon & -\alpha+\beta+\gamma-\delta & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta-\delta+\varepsilon & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta+\gamma-\varepsilon & -\alpha+\beta-\gamma+\varepsilon & \mathrm{I}-\alpha-\delta+\varepsilon & 0 \\ \alpha-\beta-\gamma+\delta & 0 & -\alpha+\beta+\gamma-\delta & \mathrm{I}-\alpha+\gamma-\varepsilon \end{vmatrix}$$

Upon subtracting the second row from the first, the new first row is

$$\mathrm{I}-\beta+\delta-\varepsilon \quad -\mathrm{I}+\beta-\delta+\varepsilon \quad -\alpha+\beta+\gamma-\delta \quad 0$$

Add the fourth column to the first and the new first column to the second.

$$\therefore D_2=\begin{vmatrix} \mathrm{I}-\beta+\delta-\varepsilon & 0 & -\alpha+\beta+\gamma-\delta & 0 \\ 0 & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta+\delta-\varepsilon & 0 & \mathrm{I}-\alpha-\delta+\varepsilon & 0 \\ \mathrm{I}-\beta+\delta-\epsilon & \mathrm{I}-\beta+\delta-\epsilon & -\alpha+\beta+\gamma-\delta & \mathrm{I}-\alpha+\gamma-\varepsilon \end{vmatrix}$$

Subtract the first row from the third and fourth rows. In the resulting determinant, subtract the second column from the fourth column. Hence

$$D_2=\begin{vmatrix} \mathrm{I}-\beta+\delta-\epsilon & 0 & -\alpha+\beta+\gamma-\delta & 0 \\ 0 & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\mathrm{I}+\beta+\gamma-\varepsilon \\ -\mathrm{I}+\alpha+\gamma-\delta & 0 & \mathrm{I}-\beta-\gamma+\epsilon & 0 \\ 0 & \mathrm{I}-\beta+\delta-\epsilon & 0 & -\alpha+\beta+\gamma-\delta \end{vmatrix}$$

By inspection, it is seen that this determinant is the square of

$$\triangle\equiv\begin{vmatrix} \mathrm{I}-\beta+\delta-\epsilon & -\alpha+\beta+\gamma-\delta \\ -\mathrm{I}+\alpha+\gamma-\delta & \mathrm{I}-\beta-\gamma+\varepsilon \end{vmatrix}$$

$$=\mathrm{I}^2+\alpha^2+\beta^2-\mathrm{I}\alpha-\mathrm{I}\beta-\alpha\beta-\gamma^2-\delta^2-\epsilon^2+\gamma\delta+\gamma\varepsilon+\delta\epsilon.$$

To show that $\triangle$ is algebraically irreducible, we note that

$$\triangle=(\mathrm{I}+\omega\alpha+\omega^2\beta)(\mathrm{I}+\omega^2\alpha+\omega\beta)-(\gamma+\omega\delta+\omega^2\epsilon)(\gamma+\omega^2\delta+\omega\varepsilon),$$

where $\omega$ is an imaginary cube root of unity. But the expressions

$$x=\mathrm{I}+\omega\alpha+\omega^2\beta,\ y=\mathrm{I}+\omega^2\alpha+\omega\beta,\ z=\gamma+\omega\delta+\omega^2\epsilon,\ w=\gamma+\omega^2\delta+\omega\varepsilon$$

are independent functions of I, $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$. Hence $\triangle\equiv xy-zw$ is irreducible.

*Hence the decomposition of the given determinant of order six into algebraically irreducible factors is as follows:**

$$D=(1+a+\beta+\gamma+\delta+\varepsilon)(1+a+\beta-\gamma-\delta-\varepsilon)\,\triangle^2.$$

In particular, $D$ is expressible as the difference of two squares,

$$D=[(1+a+\beta)\,\triangle]^2-[(\gamma+\delta+\epsilon)\,\triangle]^2.$$

The fact that $\triangle$ is a factor of $D$ may be shown directly. Multiply the second row of $D$ by $\omega$ and the third row by $\omega^2$ and add the products to the first row. Multiply the fifth row by $\omega^2$ and the sixth row by $\omega$ and add the products to the fourth row. The new first and fourth rows are

$$\begin{array}{cccccc} y & \omega y & \omega^2 y & z & \omega^2 z & \omega z, \\ w & \omega w & \omega^2 w & x & \omega^2 x & \omega x. \end{array}$$

Then develop the determinant by Laplace's method. Each term is the product of a determinant of order two formed from the above two rows by the complementary determinant of order four formed from the remaining rows. Each of these determinants of order two has the factor $xy-zw\equiv\triangle$. Hence $D$ has the factor $\triangle$.

*The University of Chicago, January, 1902.*

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ARITHMETIC.

153. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Find some two-figure numbers, such that if they be squared, then the figures interchanged and the resulting numbers squared, the resulting products will consist of the same digits in reversed order.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The difference of the two numbers is a multiple of 9.

$\therefore\ 10x+y=10y+x+9n.$

$\therefore\ x=y+n.$

Let $n=0$. The numbers are 11, 11; 22, 22.

*Since writing this paper, I find that the result was obtained in 1886 by Dedekind by the theory of matrices (Berliner Sitzungsberichte, 1897, page 1007).

Let $n=1$. The numbers are 21, 12.
Let $n=2$. The numbers are 31, 13.
Let $n=1.11$, $y=0$. The numbers are 10, .01.
Let $n=2.22$, $y=0$. The numbers are 20, .02.
Let $n=3.33$, $y=0$. The numbers are 30, .03.

Also solved by *J. H. DRUMMOND* and *J. K. ELLWOOD*.

154. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Suppose there is a meadow of 8 acres in which the grass grows uniformly, and that 21 oxen could eat up the whole pasture in 6 weeks, or 18 oxen in 9 weeks; what number of oxen diminished by the removal of 9, at the end of 14 weeks, could eat it up in 18 weeks?

I. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

In the first case, in one week one ox will eat ($\frac{1}{6}$ of $\frac{8}{21}$ of original grass + $\frac{8}{21}$ of what grows) on one acre $=\frac{4}{63}$ of original grass on one acre $+\frac{8}{21}$ of what grows on one acre.

In the second case, in one week one ox will eat ($\frac{1}{9}$ of $\frac{8}{18}$ of original grass $+\frac{8}{18}$ of what grows) on one acre $=\frac{4}{81}$ of original grass on one acre $+\frac{4}{9}$ of what grows on one acre.

In each case one ox eats the same quantity in one week.

$\therefore \frac{4}{9}-\frac{8}{21}=\frac{4}{63}$ of the growth of one acre in one week is $=\frac{4}{63}-\frac{4}{81}=\frac{8}{567}$ of an acre. $\frac{8}{567}\div\frac{4}{63}=\frac{2}{9}$ of an acre, what grows on an acre in one week. $\frac{4}{63}+\frac{8}{21}$ of $\frac{2}{9}=\frac{4}{27}$, the part of the original quantity which one ox eats in one week.

$8\div(6\times\frac{4}{27})=9$ oxen to eat original grass. $21-9=12$ oxen necessary to eat growing grass.

Now change "at the end of 14 weeks" to "at the end of 4 weeks." Otherwise we will have same hungry oxen. 8 acres $=\frac{2}{9}\times 18\times 8=40$ acres of original grass = amount on 8 acres + what grows on 8 acres in 18 weeks.

Then $4\times\frac{4}{27}$ oxen $+14\times\frac{4}{27}$ (oxen $-9$) $=40$.

$\therefore$ 72 oxen $=1080+504=1584$. $\therefore$ oxen $=22$, the number required.

Also solved by *J. R. HITT*.

II. Solution by the PROPOSER.

Let $a$ be the number of pounds of grass on 1 acre, and $an$ that which grows on 1 acre in 1 week. Then 21 oxen eat $(8a+8an\times 6)$ pounds in 6 weeks, and 18 oxen eat $(8a+8an\times 9)$ pounds in 9 weeks.

$\therefore$ From the first statement, 1 ox eats $\dfrac{4(1+6an)}{63}$ pounds in 1 week.

From the second statement, 1 ox eats $\dfrac{4(1+9an)}{81}$ pounds.

$\therefore \dfrac{4(a+9an)}{81}=\dfrac{4(a+6an)}{63}$, whence $n=\frac{2}{9}$.

Let $x$ = the required number of oxen. Then $x$ oxen will eat $\dfrac{56ax}{27}$ pounds in 14 weeks, and $(x-9)$ oxen will eat $\frac{16}{27}a(x-9)$ pounds in 4 weeks.

$\therefore \frac{56ax}{27} + \frac{16a}{27}(x-9) = 8(a+18an) = 40a$, whence $x=22$, number required.

Also solved by *J. R. HITT.*
Solutions of problem 152 were received from *J. K. ELLWOOD* and *J. R. HITT.*

## ALGEBRA.

### NOTE ON SOLUTION I. OF PROBLEM 131.

In connection with this solution it is well to note that Dr. Zerr's results hold only for the case where $a$, $b$, $c$, and $d$ are all positive, and $a^2d+b^2c-cd$ is positive, restrictions which he neglects to mention. For instance, the equation $2x-1+\sqrt{(5x^2+1)}=0$ possesses two roots, 0 and $-4$, yet $c>a^2$ or $5>2^2$.

The criteria of the second solution cover all real values of $a$, $b$, $c$, and $d$, positive or negative, and $a^2d+b^2c-cd$ positive.

H. S. VANDIVER, Bala, Pa.

133. Proposed by **HARRY S. VANDIVER**, Bala, Pa.

A theory of Fermat. The sum of two integral fourth powers cannot be an integral square. [Cf. *Chrystal's Algebra*, Vol. II, page 535.]

I. Solution by **L. C. WALKER, A. M.**, Professor of Mathematics, Petaluma High School, Petaluma Cal., and **CHAS. C. CROSS**, Whaleyville, Va.

Assume $m$ and $n$ to be the numbers, then by the condition of the problem, we have

$$(m^2)^2+(n^2)^2=c^2, \text{ suppose.....(1).}$$

In (1) either $m^2$ or $n^2$ must be even, and the other odd, because the sum of two odd squares can not be a square.

Assume $m^2$ even and $n^2$ odd.

Let $m^2=2pq$ and $n^2=p^2-q^2$; or $n^2+q^2=p^2$.....(2).

In (2) $n$ and $p$ are odd and $q$ even. Now let $q=2\alpha\beta$ and $p=\alpha^2+\beta^2$. Then we find $m^2=4\alpha\beta(\alpha^2+\beta^2)$.....(3).

Since $\alpha$ and $\beta$ are prime to each other, in order that $\alpha\beta$ may be a square, each must be a square.

Let $\alpha=m_1^2$ and $\beta=n_1^2$. Substituting in (3), we get

$$m^2=4m_1^2n_1^2(m_1^4+n_1^4).....(4).$$

In order that the right member of (4) may be a square, we must have $m_1^4+n_1^4=c_1^2$, say; which is of the same form as (1). But $c_1^2$ is less than $c^2$.

Proceeding in exactly the same way, we can reduce $c^2$ indefinitely by some integer. By our hypothesis $c^2$ can not be zero, nor less.

Hence, results *Fermat's Theorem.*

Also solved by *G. B. M. ZERR.*

II. Remarks by the PROPOSER.

Euler's proof of this theorem which appears in his treatise on Algebra, I believe to be the simplest demonstration that can possibly be obtained. For the benefit of those who are unacquainted with it, I reproduce the substance of the argument below.

EULER'S PROOF. Suppose that three integers $a$, $b$, and $c$ are found such that $a^4+b^4=c^2$. If $a$ and $b$ are not prime to each other let $a=a'y$, and $b=a'x$, then $(a'y)^4+(a'x)^4=c^2$. Whence, by division, $x^4+y^4=\square=z^2$, suppose.

Hence it is sufficient to consider the case

$$x^4+y^4=z^2 \ldots\ldots (1),$$

where $x$, $y$ and $z$ are prime to each other. If this relation be satisfied, then it follows that there are integers $p$ and $q$ such that

$$\left.\begin{array}{l} x^2=2pq \\ y^2=p^2-q^2 \end{array}\right\} \text{ or } \left\{\begin{array}{l} x^2=p^2-q^2 \\ y^2=2pq \end{array}\right.$$

Since $x$ and $y$ are symmetrical in (1) it is sufficient to put

$$x^2=2pq \ldots\ldots (2),\quad y^2=p^2-q^2 \ldots\ldots (3).$$

Since $x$ is prime to $y$ and $x$ is evidently even, it follows that $y$ is odd, and $y^2$ must then be of the form $4n+1$. In (3) we have the following hypotheses for $p^2$ and $q^2$:

$$\begin{array}{l} p^2=4m \text{ or } 4m+1 \\ q^2=4k \text{ or } 4k+1 \end{array}$$

If $p^2=4m$ and $q^2=4k$ then $y^2=4(m-k)$, which is impossible, since $y$ is odd. Put $p^2=4m$ and $q^2=4k+1$; then $y^2=4(m-k)-1$, which is also absurd.

$\therefore\ p^2=4m+1$ and $q^2=4k$, whence $p$ is odd and $q$ is even.

Therefore integers $r$ and $s$ can be found so that $p=r^2+s^2$, $q=2rs$, where $r$ and $s$ are prime to each other.

Substituting in (2) we have

$$x^2=4rs(r^2+s^2) \text{ or } x_1^2=rs(r^2+s^2).$$

Since $r$, $s$ and $r^2+s^2$ are all prime to each other we will have $r=r_1^2$, $s=s_1^2$. Then also, $r^2+s^2=r_1^4+s_1^4=\square=u_1^2$, say. Now in $r_1^4+s_1^4=u_1^2$, $r_1<r<x$.

That is, supposing (1) to hold, we have shown that it is possible to find a relation $r_1^4+s_1^4=u_1^2$ so that $r_1<x$.

In the same manner it may be shown that we can obtain $r_2^4+s_2^4=u_2^2$, where $r_2<r_1$, and evidently, $r_n^4+s_n^4=u_n^2$ where $r_n<r_{n-1}<r_{n-2}\ldots\ldots<r_1<x$.

So that by taking $n$ sufficiently large it will be found

$$1+s_n{}^4=u_n{}^2.....(4),$$

whence $u_n{}^2-s_n{}^4=1$, and $u_n=1$ and $s_n=0$ are the only solutions. Now $s_n$ cannot be 0 for if such were the case we would have

$$s_n=s_{n-1}=s_{n-2}.....=s_1=x=0,$$

which is inconsistent with the definition of $x$. Hence the impossibility of (4) and therefore of (1) is completely demonstrated.

134. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve neatly and briefly the equations

$$x^3+x^2y+y^3=53....(1),\ y^3+y^3z+z^3=13....(2),\text{ and }z^3+z^2x+x^3=31....(3).$$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Probably the only brief solution is by inspection, as follows:

$$x^3+x^2y+y^3=53=27+18+8=3^3+2.3^2+2^3.$$

$$y^3+y^2z+z^3=13=8+4+1=2^3+1.2^2+1^3.$$

$$z^3+z^2x+x^3=31=1+3+27=1^3+3.1^2+3^3.$$

$\therefore x=3,\ y=2,\ z=1.$

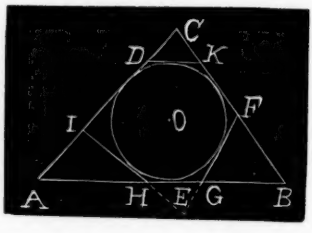
135. Proposed by CHARLES C. CROSS, Whaleyville, Va.

Tangents parallel to the three sides are drawn to the in-circle. If $p$, $q$, $r$, be the lengths of the parts of the tangents within the triangle, prove that

$$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1.$$

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let $ABC$ be any triangle with the in-circle $O$. Put $AB=c$, $AC=b$, $BC=a$.




Draw the respective tangents, $DK=r$, parallel to $AB$; $FG=q$, parallel to $AC$; and $HI=p$, parallel to $BC$.

Let $AH=x$, and $BG=y$.

The construction of lines and similarity of triangles give the following:

$HG=DE=r$; $y:c=q:b$, or $y=cq/b$;

and $x:c=p:a$, or $x=cp/a$. But $x+y+r=c$.

$$\therefore \frac{cp}{a}+\frac{cq}{b}+r=c;\text{ or }\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1.$$

Solved in a similar manner by *LON C. WALKER* and *J. SCHEFFER*.

II. Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Designate the perpendiculars from $A$, $B$, $C$ by $h'$, $h''$, $h'''$; then from similar triangles,

$$p' : a :: h' - 2r : h', \text{ or } \frac{p'}{a} = 1 - 2\frac{r}{h'}.$$

Similarly, $\frac{p''}{b} = 1 - 2\frac{r}{h''}$ and $\frac{p'''}{c} = 1 - 2\frac{r}{h'''}$; whence

$$\frac{p'}{a} + \frac{p''}{b} + \frac{p'''}{c} = 3 - 2\left(\frac{r}{h'} + \frac{r}{h''} + \frac{r}{h'''}\right) = 1 \ldots\ldots (1),$$

$$\left(\text{since, } \frac{1}{h'} + \frac{1}{h''} + \frac{1}{h'''} = \frac{1}{r}\right).$$

Similarly, if $P'$, $P''$, $P'''$, denote the length of the tangent to the *escribed* circles

$$\frac{P'}{a} + \frac{P''}{b} + \frac{P'''}{c} = 3 + 2\left(\frac{r'}{h'} + \frac{r''}{h''} + \frac{r'''}{h'''}\right) \ldots\ldots (2).$$

Again, for a *tetrahedron*, let $p'$, $p''$, $p'''$, $p''''$, designate the planes drawn tangent to the inscribed sphere and parallel to the faces $a$, $b$, $c$, $d$; and $P'$, $P''$, $P'''$, $P''''$ the plane similarly drawn tangent to the escribed sphere. Then

$$\sqrt{\frac{p'}{a}} + \sqrt{\frac{p''}{b}} + \sqrt{\frac{p'''}{c}} + \sqrt{\frac{p''''}{d}} = 4 - 2\left(\frac{r}{h'} + \frac{r}{h''} + \frac{r}{h'''} + \frac{r}{h''''}\right) = 2 \ldots\ldots (3).$$

$$\sqrt{\frac{P'}{a}} + \sqrt{\frac{P''}{b}} + \sqrt{\frac{P'''}{c}} + \sqrt{\frac{P''''}{d}} = 4 + 2\left(\frac{r'}{h'} + \frac{r''}{h''} + \frac{r'''}{h'''} + \frac{r''''}{h''''}\right) \ldots\ldots (4).$$

Also solved by *G. B. M. ZERR* and *H. C. WHITAKER*.

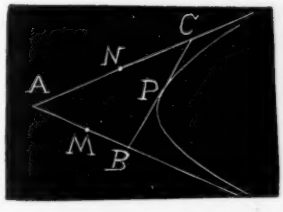
## GEOMETRY.

164. Proposed by J. M. HARCOURT, M. D., 305 Clinton Street, Brooklyn, N. Y.

Given two tangents to a parabola, find the locus of the center of the nine-point circle of the triangle by the two given tangents and any third tangent.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let the equation of the parabola be $y^2 = 4mx$; then will the equation of three tangents be of the form $y = ax + \frac{m}{a} \ldots\ldots (1)$, $y = a'x + \frac{m}{a'} \ldots\ldots (2)$, $y = a''x + \frac{m}{a''} \ldots\ldots (3)$; the equation (1) being that of $AB$, (2) of $AC$, and (3) of $BC$.




The equation of the altitude from $A$ upon

$BC$ will be

$$aa'(x+a''y)=m(1+a'a''+a''a).....(4),$$

and that of $C$ upon $AB$,

$$a'a''(x+ay)=m(1+a''a+aa'').....(5).$$

Combining (4) and (5), we find the co-ordinates of the orthocenter of $\triangle ABC$ to be $x=-m$, $y=m\left(\frac{1}{aa'a''}+\frac{1}{a}+\frac{1}{a'}+\frac{1}{a''}\right).....(6).$

The symmetry with reference to $a$, $a'$, $a''$ shows that the three altitudes all pass through a single point, and the value of $x$ shows that the locus of the orthocenter of *any* variable tangential triangle whatever is the directrix of the parabola. Since the co-ordinates of $A$, $B$, and $C$ are respectively,

$$\frac{m}{aa'},\ m\left(\frac{1}{a}+\frac{1}{a'}\right);\ \frac{m}{aa''},\ m\left(\frac{1}{a}+\frac{1}{a''}\right);\ \frac{m}{a'a''},\ m\left(\frac{1}{a'}+\frac{1}{a''}\right).....(7),$$

we find for the middle points $M$, $N$, $P$ of the sides $AB$, $AC$, $BC$, respectively, the co-ordinates

$$\frac{m}{2a}\left(\frac{1}{a'}+\frac{1}{a''}\right),\ \frac{m}{2}\left(\frac{2}{a}+\frac{1}{a'}+\frac{1}{a''}\right);\ \frac{m}{2a'}\left(\frac{1}{a}+\frac{1}{a''}\right),\ \frac{m}{2}\left(\frac{2}{a'}+\frac{1}{a}+\frac{1}{a''}\right);$$

$$\frac{m}{2a''}\left(\frac{1}{a}+\frac{1}{a'}\right),\ \frac{m}{2}\left(\frac{2}{a''}+\frac{1}{a}+\frac{1}{a'}\right).....(8).$$

Consequently the equation of the perpendicular erected at $M$ and $N$, respectively, is

$$y+\frac{1}{a}x=\frac{m}{2}\left[\frac{2}{a}+\left(1+\frac{1}{a^2}\right)\left(\frac{1}{a'}+\frac{1}{a''}\right)\right]$$

$$\text{and } y+\frac{1}{a'}x=\frac{m}{2}\left[\frac{2}{a'}+\left(1+\frac{1}{a'^2}\right)\left(\frac{1}{a}+\frac{1}{a''}\right)\right].....(9).$$

Therefore, co-ordinates of their point of intersection, or the center of the circumcircle

$$a=\frac{m}{2}\left(1+\frac{1}{aa'}+\frac{1}{aa''}+\frac{1}{a'a''}\right),\ y=\frac{m}{2}\left(\frac{1}{a}+\frac{1}{a'}+\frac{1}{a''}-\frac{1}{aa'a''}\right).....(10).$$

Since the center of the nine-point circle is the middle point of the straight line that connects the center of the circumcircle with the orthocenter, we find for the co-ordinates of the center of the nine-point circle

$$x=\frac{m}{4}\left(-1+\frac{1}{aa'}+\frac{1}{aa''}+\frac{1}{a'a''}\right),\quad y=\frac{m}{4}\left[3\left(\frac{1}{a}+\frac{1}{a'}+\frac{1}{a''}\right)+\frac{1}{aa'a''}\right]. \quad (11)$$

Considering $AB$ and $CD$ the given tangents, therefore $a$ and $a'$ given quantities, and $a''$ variable, we have in (11) only to eliminate $a''$ to obtain the equation of the required locus. Thus, putting for the sake of brevity $1/a+1/a'=a$, $1/aa'=b$, we find the equation of the locus to be

$$4ay-4(3-b)x=m[3(1+a^2)-b(4-b)].$$

Consequently, the locus is a straight line.

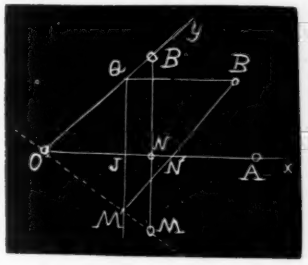
Also solved by *G. B. M. ZERR.*

165. Proposed by W. H. ECHOLS. B.Sc., S.E., Professor of Mathematics, University of Virginia, Charlottsville, Va.

$OB=b$, $OA=a$ are the semi-conjugate diameters of an ellipse. Draw $BM$ perpendicular to and equal to $OA$, cutting it in $N$. Show that as $M$ slides on the fixed line $OM$ and $N$ on $OA$ the point $B$ traces the curve.

Solution by the PROPOSER.

Let $M'N'B'$ be an arbitrary position of the line. Draw $B'Q$ parallel to $Ox$, join $M'Q$. Then




$$\frac{M'J}{JQ}=\frac{M'N'}{N'B'}=\frac{MN}{NB}.$$

$\therefore$ $M'Q$ is perpendicular to $Ox$.
Hence, if $B'Q=x$, $OQ=y$, we have

$$x^2+(M'Q)^2=a^2, \text{ or } x^2+\frac{a^2}{b^2}y^2=a^2.$$

$$\therefore\ x^2/a^2+y^2/b^2=1. \qquad \text{Q. E. D.}$$

166. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Two cities are 200 miles apart. To what height must a man ascend from one city in order that he may see the other, supposing the circumference of the earth to be 25,000 miles? [From Wentworth's *New Plane and Solid Geometry*, page 381, No. 619.] Required solution by Geometry.

Solution by DANIEL NORTHROP, Mandana, N. Y.

Let $A$ be the position of one city and $B$ the position of the second, distant on a straight line from $A$, $a=200$ miles. Let $P$ be the position of the man above $A$ when just able to see the city $B$, $h$ his height above $A$. Draw the line $PA$ and extend it through the center of the earth to the point $D$ opposite $A$. Draw the line $PB$. Then we have $PB^2=PD\times PA$, or $PB^2=(h+2R)h$.

From the similar triangles $PAB$ and $PBD$, we have $PB:DB=PA:AB$, or $PB:DB=h:a$. But $DB=\sqrt{(AD^2-AB^2)}=\sqrt{(4R^2-a^2)}$.

$\therefore PB:\sqrt{(4R^2-a^2)}=h:a$. Whence $PB=\dfrac{h\sqrt{(4R^2-a^2)}}{a}$.

$$\therefore \frac{h^2(4R^2-a^2)}{a^2}=h(h+2R).$$

$$\therefore h=\frac{2Ra^2}{4R^2-2a^2}=\frac{Ra^2}{2R^2-a^2}=5.0062 \text{ miles.}$$

Also solved by *G. B. M. ZERR, L. C. WALKER, H. C. WHITAKER,* and *J. SCHEFFER.*

## CALCULUS.

127. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg. Pa.

Find the moment of inertia of a parallelogram about an axis perpendicular to its plane and passing through the intersection of its diagonals.

Solution by G. B. M. ZERR, A.M., Ph. D., The Temple College, Philadelphia, Pa.; J. SCHEFFER, A. M., Hagerstown, Md., and J. M. ARNOLD, Crompton, R. I.

Let the intersection of the diagonals be the origin; straight lines through the origin parallel to the sides, the axes; $a$, $b$ the sides.

Then the required moment of inertia is

$$I=\rho\sin\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}[(x+y\cos\beta)^2+y^2\sin^2\beta]dxdy$$

where $\beta$ is the acute angle of the parallelogram.

$$\therefore I=\rho\sin\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}(bx^2+\tfrac{1}{12}b^3)dx=\tfrac{1}{12}\rho ab\sin\beta(a^2+b^2)=\tfrac{1}{12}m(a^2+b^2).$$

Also solved by *W. J. GREENSTREET* and *L. C. WALKER.*

128. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The differential equation of a curve is $\dfrac{d^3y}{dx^3}+y=0$. Find its equation, there being the additional conditions that for $x=0$, $y=1$, that the tangent at the point (0, 1) makes an angle of 45° with the axes, and finally that that point is a point of inflexion.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.; and L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Using $D$ for the operator $d/dx$, the given equation is

$$(D^3+1)y=0.....(1), \text{ or } (D^2-D+1)(D+1)y=0.....(2).$$

Integrating $(D+1)y=e^{\frac{1}{2}x}(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x)$.....(3).

Differentiating this with respect to $x$,

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=e^{\frac{1}{2}x}[(\tfrac{1}{2}\cos\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}x)A$$

$$+(\tfrac{1}{2}\sin\frac{\sqrt{3}}{2}x+\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x)B]\text{.....(4).}$$

When $x=0$, $y=1$, $dy/dx=1$, and (3) gives $A=2$.
When $x=0$, $D=1$, $D^2=0$, and (4) gives $B=0$.

$\therefore$ (3) is $(D+1)y=2e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x$.....(5).

Integrating (5), and noticing that when $x=0$, and $y=1$, the constant of integration is nought,

$$y=e^{\frac{1}{2}x}(\tfrac{1}{3}\sqrt{3}\sin\frac{\sqrt{3}}{2}x+\cos\frac{\sqrt{3}}{2}x)\text{.....(6).}$$

Also solved by *G. B. M. ZERR* and *J. SCHEFFER*.

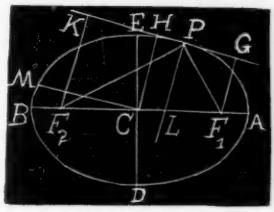
## MECHANICS.

**128. Proposed by M. E. GRABER, A. B., Heidelberg University, Tiffin, Ohio.**

A particle is placed on the convex side of a smooth ellipse and is acted upon by two forces, $F$ and $F'$, towards the foci, and a force, $F''$, towards the center. Find the position of equilibrium.

**Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and the PROPOSER.**

Let $C$ be the center of the ellipse, semi-axes $a$, $b$; $F_1$, $F_2$, the foci; force $PF_1=F$, $PF_2=F'$, $PC=F''$; distance $PC=r$. Let fall the perpendiculars $F_1G$, $CH$, $F_2K$ from the foci $F_1$, $F_2$, and the center $C$ on the tangent at $P$.




Let $\angle PF_1G=\angle PF_2K=\theta$, $\angle PCH=\varphi$.

Then $F\sin\theta=F'\sin\theta+F''\sin\varphi$.

$\therefore \dfrac{\sin\varphi}{\sin\theta}=\dfrac{F-F'}{F''}=n$, suppose.

$F_2K=F_2P\cos\theta$, $F_1G=F_1P\cos\theta$, $CH=r\cos\varphi$.

$2CH=F_2K+F_1G=(F_2P+F_1P)\cos\theta=2a\cos\theta$.

$\therefore r\cos\varphi=a\cos\theta$.

$F_2K.F_1G=b^2=F_2P.F_1P\cos^2\theta=CM^2\cos^2\theta=(a^2+b^2-r^2)\cos^2\theta$.

$$\therefore \cos\theta=\frac{b}{\sqrt{(a^2+b^2-r^2)}},\quad \sin\theta=\frac{\sqrt{(a^2-r^2)}}{\sqrt{(a^2+b^2-r^2)}}.$$

$$\therefore \cos\varphi = \frac{ab}{r_1\sqrt{(a^2+b^2-r^2)}}, \quad \sin\varphi = \frac{\sqrt{[r^2(a^2+b^2-r^2)-a^2b^2]}}{r_1\sqrt{(a^2+b^2-r^2)}}.$$

$$\therefore n = \frac{\sqrt{[r^2(a^2+b^2-r^2)-a^2b^2]}}{r_1\sqrt{(a^2-r^2)}} = \frac{\sqrt{(r^2-b^2)}}{r}.$$

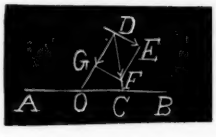
$$\therefore r^2-b^2 = r^2n^2, \text{ or } r = \frac{b}{\sqrt{(1-n^2)}}.$$

129. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Two spheres whose masses are $M_1$ and $M_2$ are $a$ units apart, and attract each other with a force $= M_1M_2/a_2$. Find work done in carrying a unit mass from the center point between them a distance $r$ in a direction $\theta$ with line of centers.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $B = m_1$, $A = m_2$, $m_1 > m_2$; $C$, the centroid of $m_1$, $m_2$; $OC = c$, $OD = x$, $DC = y$, $\angle DOC = \theta$, $\angle OCD = \varphi$.




Then $y = \sqrt{(c^2+x^2-2cx\cos\theta)}$, $c = \frac{a(m_1-m_2)}{2(m_1+m_2)}$.

$c = y\cos\varphi + x\cos\theta$. $\quad\therefore \cos\varphi = \frac{c-x\cos\theta}{y}$.

Force acting on $D = (m_1+m_2)/y^2$.

Resolving this force, the component along $OD$

$$= (m_1+m_2)\cos(\theta+\varphi)/y_2 = \frac{m_1+m_2}{y^3}\{(c-x\cos\theta)\cos\theta - \sin\theta\sqrt{[y^2-(c-x\cos\theta)^2]}\}$$

$$= \frac{(m_1+m_2)(c\cos\theta-x)}{(c^2+x^2-2cx\cos\theta)^{\frac{3}{2}}}.$$

$$\text{Work} = (m_1+m_2)\int_0^r \frac{(c\cos\theta-x)dx}{(c^2+x^2-2cx\cos\theta)^{\frac{3}{2}}}$$

$$= (m_1+m_2)\left(\frac{1}{\sqrt{(c^2+r^2-2cr\cos\theta)}} - \frac{1}{c}\right)$$

$$= 2(m_1+m_2)^2\left(-\frac{1}{\sqrt{4r^2(m_1+m_2)^2 + a^2(m_1-m_2)^2 - 4ar(m_1^2-m_2^2)\cos\theta}}\right.$$

$$\left. - \frac{1}{m_1-m_2}\right).$$

## DIOPHANTINE ANALYSIS.

86. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Prove that the congruence $x^2=1457\equiv 0 \pmod{2389}$ is not possible.

A solution of this problem is given in Mathews' *Theory of Numbers*, page 42. ED.

87. Proposed by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Find three numbers in arithmetical progression the sum of whose cubes is a cube.

Solution by the PROPOSER.

Let $tx$, $ty$, $tz$ represent the three required numbers. Then we have

$$t^3x^3+t^3y^3+t^3z^3=w^3;\ \text{or, } x^3+y^3+z^3=w^3/t^3=v^3\ldots\ldots(1),\ \text{assume.}$$

Transposing (1), we have to satisfy the equation

$$v^3-x^3-y^3=z^3\ldots\ldots(2).$$

Put $x=a(r-q)$, $y=aq$, $z=a[p-(rq^2/p^2)]$, and $v=ap$. Then we have, after dividing by $a^3$,

$$p^3-(r-q)^3-q^3=(p-\frac{rq^2}{p^2})^3\ldots\ldots(3).$$

Whence, by involution and reduction,

$$r=\frac{3p^3q}{p^3+q^3}.$$

Now take $a=p^3+q^3$, and we have

$$x=q(2p^3-q^3),\ y=q(p^3+q^3),\ z=p(p^3-2q^3),\ v=p(p^3+q^3),$$

where $p$ and $q$ may have any values subject to the condition $p^3>2q^3$.

If we take $p=2$, $q=1$, after dividing all by 3, we have, $3^3+4^3+5^3=6^3$, the least three positive integral numbers that satisfy conditions of the problem.

[See *Mathematical Magazine*, page 154, from whence the above solution was copied.]

88. Proposed by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Find three square numbers in harmonical progression.

Solution by HARRY S. VANDIVER, Bala, Pa.

Solutions of this problem have been given on pages 82-83 of Vol. VII, of the MONTHLY, substantially as follows:

Let $1/a^2$, $1/b^2$, $1/c^2$, be the numbers. Then $a^2+c^2=2b^2$, and solutions are given by

$$a=m^2-n^2+2mn$$
$$c=n^2-m^2+2mn$$
$$b=m^2+n^2.$$

To make these solutions complete, however, it is necessary to show that the formulae for $a$, $c$, and $b$ give all the possible solutions. This may be proved as follows:

If $a^2+c^2=2b^2$ then $b$ must be of the form $m^2+n^2$, for every divisor of the sum of two squares is itself the sum of two squares. Now

$$2b^2=2(m^2+n^2)^2=(1+1)(m^2+n^2)^2$$

can be expressed as the sum of two squares in but one way, the following:

$$(m^2-n^2-2mn)^2+(m^2-n^2+2mn).$$

Hence $a=m^2-n^2+2mn$, and $c=n^2-m^2+2mn$ give all the solutions of the problem.

Also solved by *L. C. WALKER*.

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## AVERAGE AND PROBABILITY.

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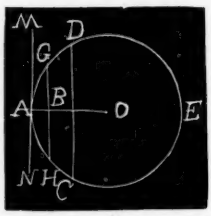
109. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A cylinder pierces a sphere in such a manner that the cylinder is tangent internally to the projection of the sphere in the plane $xy$. Find (1) the average surface, (2) the average volume of the sphere included within the cylinder.

Solution by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Let $ACED$ be a section of the sphere through the center and the axis of the cylinder, $O$ the center of the sphere, $GH$ the axis of the cylinder, $MN$ an element of the cylinder tangent to the sphere at $A$. Let $S$, $V$, be the required average surface and volume, respectively.




Put $OA=a$, $AB=r$. Now if we take $AO$ for the axis of $x$, $AM$ for the axis of $z$, and the perpendicular to $OAM$ at $A$ for the axis of $y$, the equations of the sphere and cylinder are respectively,

$$x^2+y^2+z^2=2ax, \text{ and } x^2+y^2=2rx.$$

$$\therefore\ S=\frac{2\int_0^a\int_0^{2r}\int_0^{\sqrt{(2rx-x^2)}}\frac{a\,dr\,dx\,dy}{\sqrt{(2ax-x^2-y^2)}}}{\int_0^a dr}=\pi\int_0^a\int_0^{2r}dr\,dx=\pi a^2,$$

the area of a great circle of the sphere.

If $y'=\sqrt{(2ax-x^2)}$, $z=\sqrt{(2ax-x^2-y^2)}$, then we have

$$V=\frac{4}{a}\int_0^a\int_0^{2r}\int_0^{y'}\int_0^{z'}dr\,dx\,dy\,dz=\frac{4}{a}\int_0^a\int_0^{2r}\int_0^{y'}(2ax-x^2-y^2)\,dr\,dx\,dy$$

$$=\frac{2}{a}\int_0^a\int_0^{2r}\left[(2ax-x^2)\sin^{-1}\left(\frac{2r-x}{2a-x}\right)^{\frac{1}{2}}+x\sqrt{[2(a-r)(2r-x)]}\right]dr\,dx$$

$$=\frac{8}{9a}\int_0^a\left[3a^3\tan^{-1}\left(\frac{r}{a-r}\right)^{\frac{1}{2}}-(3a^2+2ar-8r^2)\sqrt{(ar-r^2)}\right]dr=\tfrac{5}{6}a^3\pi$$

$=\frac{5}{8}$ of the volume of the sphere.

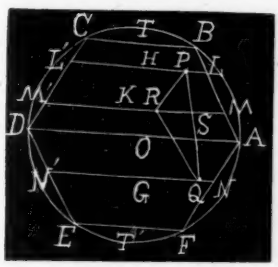
Also solved by *G. B. M. ZERR*, who gets $\frac{1}{2}\pi R^2$ as a result for the second part of the problem.
A partial solution was received from F. P. Matz.
Professor Walker should have received credit in the last issue for solution of problem 108.

110. Proposed by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the average area of the triangle formed by joining three random points taken on the surface of a regular hexagon, two on one side of a diagonal and the third on the other side.

Solution by the PROPOSER.

Let $ABCDEF$ be the hexagon; $P$, $R$ the random points above the diagonal $AD$; $Q$ the random point below the diagonal. Through $P$, $R$, $Q$ draw $LL'$, $MM'$, $NN'$ parallel to $AD$, and $TT'$ perpendicular to $AD$ through $O$. It is only necessary to consider the relative positions in which the line $MM'$ lies between $LL'$ and $NN'$.




Let $CB=OA=a$ be the side of the hexagon, $OH=u$, $OG=v$, $OK=w$, $HP=x$, $GQ=y$, $KR=z$, $KS=t$, $HL=x'$, $GN=y'$, $KM=z'$.

Then $OT=\frac{1}{2}a\sqrt{3}=u'$, $x=(a\sqrt{3}-u)/\sqrt{3}$, $y'=(a\sqrt{3}-v)/\sqrt{3}$, $z'=(a\sqrt{3}-w)/\sqrt{3}$,

$$t=y-[(y-x)(v+w)]/(u+v).$$

Area $PQR=\frac{1}{2}(t-z)(u+v)=A$, when $t>z$.

Area $PQR=\frac{1}{2}(z-t)(u+v)=A_1$, when $t<z$.

The limits of $u$ are 0 and $\frac{1}{2}a\sqrt{3}$; of $v$, 0 and $\frac{1}{2}a\sqrt{3}$; of $w$, 0 and $u$; of $x$, $-x'$ and $x'$; of $y$, $-y'$ and $y'$; of $z$, $-z'$ and $t$, and $t$ and $z'$.

The whole number of ways the three points can be taken is $\frac{81\sqrt{3}}{64}a^6$. Doubling, since the halves are interchangeable, we get for average area of triangle:

$$\triangle = \frac{128}{81\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{u}\int_{-x'}^{x'}\int_{-y'}^{y'}\left[\int_{-z'}^{t}A\,dz+\int_{t}^{z'}A_1\,dz\right]du\,dv\,dw\,dx\,dy$$

$$= \frac{64}{81\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{u}\int_{-x'}^{x'}\int_{-y'}^{y'}\left[\tfrac{1}{3}(a\sqrt{3}-w)^2+\left(y-\frac{(y-x)(v+w)}{u+v}\right)^2\right]$$

$$\times(u+v)\,du\,dv\,dw\,dx\,dy$$

$$= (\tfrac{2}{3})^7\frac{1}{a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{u}\int_{-x'}^{x'}\left[3(a\sqrt{3}-w)^2(a\sqrt{3}-v)(u+v)\right.$$

$$\left.+\frac{(a\sqrt{3}-v)^3(u-w)^2+9x^2(a\sqrt{3}-v)\,(v+w)^2}{u+v}\right]du\,dv\,dw\,dx$$

$$= (\tfrac{2}{3})^8\frac{\sqrt{3}}{a^5}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{u}\left[3(a\sqrt{3}-u)(a\sqrt{3}-v)(a\sqrt{3}-w)^2(u+v)\right.$$

$$\left.+\frac{(a\sqrt{3}-u)(a\sqrt{3}-v)^3(u-w)^2+(a\sqrt{3}-u)^3(a\sqrt{3}-v)\,(v+w)^2}{u+v}\right]du\,dv\,dw$$

$$= (\tfrac{2}{3})^8\frac{1}{\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\left[9a^3\sqrt{3}(a\sqrt{3}-u)(a\sqrt{3}-v)(u+v)\right.$$

$$-3(a\sqrt{3}-u)^4(a\sqrt{3}-v)(u+v)+(a\sqrt{3}-u)^3(a\sqrt{3}-v)(u+v)^2$$

$$\left.+\frac{u^3(a\sqrt{3}-u)(a\sqrt{3}-v)^3-v^3(a\sqrt{3}-u)^3(a\sqrt{3}-v)}{u+v}\right]du\,dv$$

$$= (\tfrac{2}{3})^5\frac{1}{27\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\left[8\sqrt{3}au^6-42a^2u^5+127\sqrt{3}a^3u^4-480a^4u^3+81\sqrt{3}a^5u^2\right.$$

$$\left.+216a^6u+16u^3(9a^4-u^4)\log\left(\frac{2u+a\sqrt{3}}{2u}\right)\right]du = \frac{1507\sqrt{3}}{11664}a^2.$$

## MISCELLANEOUS.

104. Proposed by HARRY S. VANDIVER, Bala, Pa.

A Theorem of Fermat. The area of a right angled triangle with commensurable sides cannot be a square number. [Cf. Chrystal's *Algebra*, Vol. II., page 535.]

II. Solution by the PROPOSER.

Analytically expressed, Fermat's assertion is equivalent to the following: No positive integers can be found to satisfy the relation,

$$xy(x^2-y^2)=z^2 \ldots\ldots (1).$$

To prove this suppose that there are three integers, $x_0$, $y_0$, $z_0$, such that

$$x_0y_0(x_0^2-y_0^2)=z_0^2 \ldots\ldots (2).$$

Assume that $x_0$ and $y_0$ have a common factor, then we may put $x_0=pa$, and $y_0=pb$. Then substituting, $abp^4(a^2-b^2)=z_0^2$, or by division $ab(a^2-b^2)=\square=u^2$, say (where $a$ and $b$ are prime to each other). Hence it is sufficient to prove the impossibility of (2) only in the case when $x_0$ and $y_0$ are prime to each other.

The form $ax_1^2$ where $a=p\times q\times r\ldots\ldots$ ($p$, $q$, $r$, etc., being primes) and where either $a$ or $x_1$ may become 1, represents all values of $x_0$. Putting it in this form, then the quantity $a$ must occur as a factor in $y_0(x_0^2-y_0^2)$ since $x_0y_0(x_0^2-y_0^2)$ is a square number, that is, one of the three following relations must be satisfied:

$$y_0\equiv 0 (\mathrm{mod}\, a)$$
$$x_0-y_0\equiv 0 (\mathrm{mod}\, a)$$
$$x_0+y_0\equiv 0 (\mathrm{mod}\, a)$$

Unless $a=1$, each of these relations show that $x_0$ and $y_0$ have a common factor, a result contrary to hypothesis.

Hence $x_0=x_1^2$, and it may be proved in a similar manner that $y_0=y_1^2$.

Substituting these values in (2) we have

$$x_1^2y_1^2(x_1^4-y_1^4)=z_0^2, \text{ whence } x_1^4-y_1^4=\square=z_1^2, \text{ say.}$$

Hence we infer, that if (1) is to be satisfied by integral values of $x$, $y$, and $z$ it must be possible to find integers $x_1$, $y_1$, $z_1$ such that

$$x_1^4-y_1^4=z_1^2 \ldots\ldots (3).$$

But it may be shown that there are no such integers $x_1$, $y_1$, $z_1$.

For, consider the expression $x_1^2+y_1^2$. Its general form is $bx_2^2$ where $b$ is the product of unequal primes, that is,

$$bx_2^2=x_1^2+y_1^2.$$

From this, $x_1^2+y_1^2\equiv 0 (\mathrm{mod}\, b)$

and from (3), $x_1^2-y_1^2\equiv 0 (\mathrm{mod}\, b)$

whence $2x_1^2\equiv 0 (\mathrm{mod}\, b)$

and $2y_1^2\equiv 0 (\mathrm{mod}\, b)$

and since $x_1$ and $y_1$ are prime to each other, we see that $b$ must have either of the values 1 and 2; that is, either of the two following sets of relations must be satisfied:

$$1\text{st}\begin{cases} x_1^2+y_1^2=x_2^2 \\ x_1^2-y_1^2=y_1^2 \end{cases} \qquad 2\text{nd}\begin{cases} x_1^2+y_1^2=2x_2^2 \\ x_1^2-y_1^2=2y_2^2 \end{cases}$$

Take the first set, writing it as follows:

$$x_1^2-y_1^2=y_2^2 \ldots\ldots(4),$$
$$x_2^2-y_1^2=x_1^2 \ldots\ldots(5).$$

Then proceeding as we did with $x_1^4-y_1^4=z_1^2$, it can be shown that two of the following four sets must hold:

$$\text{I}\begin{cases} x_1+y_1=m^2 \\ x_1-y_1=n^2 \end{cases} \qquad \text{II}\begin{cases} x_2+y_1=r^2 \\ x_2-y_1=s^2 \end{cases} \quad (rs=x_1)$$

$$\text{III}\begin{cases} x_1+y_1=2m^2 \\ x_1-y_1=2n^2 \end{cases} \qquad \text{IV}\begin{cases} x_2+y_1=2r^2 \\ x_2-y_1=2s^2 \end{cases}$$

From (4) and (5) it will be seen that $x_1$ and $x_2$ are odd and $y_1$ is even. Therefore sets III and IV cannot be true. Using the first two and transforming

$$x_1=\frac{m^2+n^2}{2}=rs, \text{ or } y_1=\frac{m^2-n^2}{2}=\frac{r^2-s^2}{2}$$

$$\text{or } m^2+n^2=2rs \ldots\ldots(6),$$
$$m^2-n^2=r^2-s^2 \ldots\ldots(7).$$

Then it is *necessary* that there be integers $a$, $b$, $l$, $k$ such that $r-s=ak$, $r+s=bl$ (any of the quantities $a$, $b$, $l$, $k$ may be $=1$).

Then from (7), $m-n=ab$, $m+n=kl$.

Substituting in (6) the values found for $r$, $s$, $m$, and $n$ from these relations, and reducing,

$$a^2b^2+k^2l^2+a^2k^2-b^2l^2=0,$$
whence $$a^2(b^2+k^2)=l^2(b^2-k^2),$$
and therefore $$b^4-k^4=\square \ldots\ldots(8).$$

Now since $x_1=\frac{m^2+n^2}{2}$, and also $ab=m-n$, and $\frac{m^2+n^2}{2}>m-n$,

we have $$x_1>ab,$$
whence $$x_1>b.$$

Hence, assuming $x_1^2+y_1^2=x_2^2$ and $x_1^2-y_1^2=y_2^2$
we have shown that if (3) is satisfied by $x_1$, $y_1$, $z_1$, then it follows that there is a relation $b^4-k^4=\square$ where $b<x_1$. Using the second assumption, and the only other possible one, namely,

$$\begin{cases} x_1^2+y_1^2=2x_2^2 \\ x_1^2-y_1^2=2y_2^2 \end{cases}$$

we obtain, by addition and subtraction,

$$x_2^2+y_2^2=x_1^2 \ldots\ldots (9),$$
$$x_2^2-y_2^2=y_1^2 \ldots\ldots (10),$$

and multiplying, $x_2^4-y_2^4=\square$. But (9) gives $x_2<x_1$.

Hence it has been shown that using this assumption, we can obtain from (3) a relation $x_2^4-y_2^4=\square$ such that $x_2<x_1$. Therefore, if (3) holds it is *always* possible to find a relation $x_2^4-y_2^4=z_2^2$ such that $x_2<x_1$.

From $x_2^4-y_2^4=z_2^2$ it is then possible to find $x_3^4-y_3^4=z_3^2$ where $x_3<x_2$, and so we can get any number of equations, of the type, $x_n^4-y_n^4=z_n^2$, where

$$x_n<x_{n-1}<x_{n-2}\ldots\ldots<x_3<x_2<x_1.$$

Since $x_n$ is always positive, by taking $n$ sufficiently large, there is obtained $1-y_n^4=z_n^2$, which is impossible for positive integral values of $y_n$ and $z_n$.

Hence the impossibility of $x_1^4-y_1^4=z_1^2$ has been completely established, and therefore the fact that $xy(x^2-y^2)=z^2$ cannot be satisfied, follows.

**105.** Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

If the refractive index of a medium at any point be $\mu=x$, prove that the path of the ray will be the curve $\frac{2x}{a}=\frac{c}{a}e^{y/a}+\frac{a}{c}e^{-(y/a)}$, $a$ and $c$ being constants.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let $(x, y)$ be any point in the path; $\mu=kx$, the index of refraction; and $p=dy/dx$. The differential equation to the path is

$$\frac{dp/dx}{1+p^2}=\frac{1}{\mu}\left(\frac{d\mu}{dy}-\frac{d\mu}{dx}\frac{dy}{dx}\right)\ldots\ldots(1).$$

$d\mu/dx=k$, and (1) reduces to

$$\frac{dp}{p(1+p^2)}=-\frac{dx}{x}\ldots\ldots(2).$$

Integrating, $\log\frac{px}{\sqrt{(1+p^2)}}=C=\log a$, say.....(3).

This gives $p=\frac{dy}{dx}=\frac{a}{\sqrt{(x^2-a^2)}}$.....(4).

Integrating (4), $y=a\log[x+\sqrt{(x^2-a^2)}]+C$.....(5).

Let $y=0$, when $x=b$; then $C=-a\log[b+\sqrt{(b^2-a^2)}$, and (5) becomes after putting $c=b+\sqrt{(b^2-a^2)}$,

$$\frac{2x}{a}=\frac{c}{a}e^{y/a}+\frac{a}{c}e^{-(y/a)}.....(6),$$

$e$ being the Naperian base.

Also solved by *G. B. M. ZERR* and *L. C. WALKER*.

106. Proposed by J. W. YOUNG, Oliver Graduate Student in Mathematics, Cornell University, Ithaca, N. Y.

Prove that $\frac{(2m)!}{(m!)^2}$ is an integer; and more generally that $\frac{(nm)!}{(m!)^n}$ is an integer; $m$, $n$ being any positive integers.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

It has been demonstrated that the product of any $n$ successive integers is divisible by $n!$

$$\frac{(nm)!}{(m!)^n}\div\frac{[(n-1)m]!}{(m!)^{n-1}}=\frac{(nm)!}{[(n-1)m]!}\cdot\frac{1}{m!}=\frac{(nm)!}{(nm-m)!}\cdot\frac{1}{m!}$$

$$=\frac{nm(nm-1)(nm-2).....\text{to } m \text{ factors}}{m!}=\text{an integer.}$$

$\therefore$ If $\frac{[(n-1)m]!}{(m!)^{n-1}}$ is an integer, so is $\frac{(nm)!}{(m!)^n}$.

But $\frac{m!}{m!}$ is an integer. $\therefore$ $\frac{(2m)!}{(m!)^2}$ is an integer, and so on to $\frac{(nm)!}{(m!)^n}$.

Also solved by *H. S. VANDIVER*.

# PROBLEMS FOR SOLUTION.

## ARITHMETIC.

156. Proposed by JAMES F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

Suppose that in a meadow the grass is of uniform quality and growth, and that 6 oxen or 10 colts could eat up 3 acres of pasture in 18-25 of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require 2 6-7 weeks longer than 660 sheep to eat 9 acres. In what time could 1 ox, 1 colt, and 1 sheep eat up 1 acre of pasture, on the supposition that 588 sheep eat as much in a week as 6 oxen and 11 colts?

157. Proposed by B.F.FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

January 1, 1899, A and B entered into partnership for 3 years. A put in $10,000 and B put in $5,500. July 1, 1899, B put in $1,500 more. October 1, A took out $500. January 1, 1900, each put in $1,500. July 1, 1900, they dissolved partnership, and found that they had lost $846. What is each partner's share of the loss?

## ALGEBRA.

159. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

If $x-1=3m$, $x^2-1=4n$, $x^3-1=5p$, where $m$, $n$, $p$ are integers, find a general expression for $x$.

151. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Represent the square root of $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$ as the sum of three square roots.

152. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

If $n$ quantities are made up of $q$ sets of $r$ each, find the number of permutations $s$ at a time. It is supposed that the quantities in each set are alike, but different from those in the other sets.

## GEOMETRY.

184. Proposed by ERWIN MARTIN, Principal of Schools, Mead, Neb.

If from any point in the circumference of a circle circumscribed about a triangle, perpendiculars are drawn to the sides, or the sides produced, of the inscribed triangle, the lines connecting the feet of the perpendiculars are collinear.

185. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Given the tangential equations to two conics $S$, $S'$, find the tangential co-ordinates of the join of the poles of two given parallel lines with respect to $S$. Deduce the tangential equation of the center of $S$, and find that of the intersection of $S$ and $S'$.

186. Proposed by J. R. HITT, Professor of Mathematics, San Marcos, Tex.

If two sides of a triangle and its in-circle be given in position, the envelope of its circum-circle is a circle (*Mannheim*). [From Casey's *Sequel to Euclid.*]

## CALCULUS.

150. Proposed by E. B. ESCOTT, Instructor in Mathematics, University of Michigan, Ann Arbor, Mich.

Find total area between the curve $x^4y-x^2+4y-1=0$ and the $x$-axis.

151. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Integrate the differential equation: $xy\frac{\partial^2 z}{\partial x.\,\partial y}=bx\frac{\partial z}{\partial x}+ag.$

## MECHANICS.

140. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

A long row of particles, each mass $m$, is placed on a smooth horizontal table. Each is connected with the two adjacent ones by similar light elastic strings of natural length $l$. They receive small longitudinal disturbances such that each of them proceeds to perform a harmonic oscillation. Prove that there will be two waves of vibration in opposite directions with the same velocity, viz, $l'\sqrt{\frac{E}{ml}}\frac{q}{\pi}\sin\frac{\pi}{q}$, when $l'$ is the average distance between two successive particles, $q$ the number of intervals between two particles in the same phase, and $E$ the modulus of elasticity. [*Mathematical Tripos*, 1873.]

141. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

A simple pendulum hangs from a bicycle moving in a straight line. What deflection is produced by putting on the brake so as to exert on the machine a force equal to the $n$th of its weight?

## DIOPHANTINE ANALYSIS.

101. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that it is impossible to find integral values for $x$, $y$, and $z$ such that the relation $x^2y+xz^2=y^2z$ is satisfied.

N. B. Problems in this department beginning with December number should be renumbered; those in December number being 96 and 97, respectively; January number, 98 and 99; February number, 100.

## AVERAGE AND PROBABILITY.

125. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

A circle with unknown radius is described with its center at the extremity of the major axis of a given ellipse. Required the average area common to the circle and the ellipse.

126. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the average ellipse inscribed in a triangle, so that the sides of the triangle are tangent to the ellipse.

## MISCELLANEOUS.

126. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The declination of a certain fixed star is 12° 40′. Its altitude was observed one day to be 16° 40′. Three hours and twenty-four minutes later it was found to be 40° 20′. Find the latitude of the place of observation.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Show how to calculate the velocity of an electrical discharge between a cloud and the earth. Is this velocity a function of the quantity of electricity discharged?

## *BOOKS AND PERIODICALS.*

*Elementary Treatise on Navigation and Nautical Astronomy.* By Eugene L. Richards, M. A., Professor of Mathematics in Yale University. 16mo. Cloth, 173 pages. Price, $1.00. New York and Chicago: The American Book Co.

This is a neat little work dealing with some interesting problems in Navigation. It is elementary and clear in its treatment of subjects, and can therefore be easily read by any one having a knowledge of geometry and trigonometry.

*Elements of Plane Geometry.* By Alan Sanders, Hughes High School, Cincinnati, Ohio. 8vo. Cloth. Price, $1.00. New York and Chicago: The American Book Co.

This book does not differ very essentially from a great number of recent text-books on geometry. It contains a large list of well-selected exercises for original work on the part of the student, and the demonstrations and suggestions are good. The mechanical make-up of the book is first-class.

*Vector Analysis.* A Text-book for the Use of Students of Mathematical Physics. Founded upon the Lectures of J. Willard Gibbs, Ph. D., LL. D., Professor of Mathematical Physics in Yale University. By Edwin Bidwell Wilson, Ph. D., Instructor in Mathematics in Yale University. 8vo. Cloth, xviii+436 pages. Price, $4.00, net. New York: Charles Scribner's Sons.

This work belongs to that series of publications known as the "Yale Bicentennial Publications," and is the largest treatise thus far written on this subject in America. The body of the work is divided into six chapters of which the first treats of Addition and Scalar Multiplication; the second of Direct and Akew Products of Vectors; the third of Differential Calculus of Vectors; the fourth of Integral Calculus of Vectors; the fifth of Linear Vector Functions; and the sixth of Rotations and Strains; chapter seven deals with Miscellaneous Applications. This work will be of great value to the mathematician as well as the mathematical physicist.

*An Elementary Book on Electricity and Magnetism and Their Applications.* A text-book for Manual Training Schools and High Schools, and a Manual for Artesans, Apprentices, and Home Readers. By Dugal C. Jackson, C. E., Professor of Electrical Engineering, University of Wisconsin, member of the American Institute of Electrical Engineering, etc.; and John Price Jackson, M. E., Professor of Electrical Engineering, Pennsylvania State College, member of the American Institute of Electrical Engineering, etc. 16mo, cloth sides and leather back, xi+482 pages. New York: The Macmillan Co.

This is the best work of its kind that has yet appeared. It treats in a very lucid manner, every department of electricity and magnetism and their various applications. While written in somewhat untechnical language, yet it is clear, forceful, and scientifically accurate. In this work is found a practical treatment of the dynamo, motor, telegraph, telephone, electric lighting, electric smelting, welding, cooking, wireless telegraphy, Roentgen rays, and many other subjects. Here is a complete description of Niagara Falls Power Company's plant, with illustrations and a map, covering six pages. Throughout the book, frequent use of analogies are made, thus enabling the reader to obtain the clearest possible conceptions of electric and magnetic properties.

In the preparation of this book, the authors have contributed a most valuable work to the literature of the subject, and every one interested in the extension of knowledge and the enlarging of social conditions will want ta have a copy of the book.

*The American Monthly Review of Reviews.* An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

*The Literary Digest.* A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

*The Cosmopolitan.* In International Illustrated Monthly Magazine. Edited and Published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

ERRATA.

Page 34, first line under the title, for "1809" read 1896, 3rd edition; and for "Sir" read W.

Page 41, fourth line of second solution, for "rational" read real.

Page 41, ninth line of second solution, for "$\pm$" read $=$.

Page 43, in the last line but three of the first demonstration in Geometry, for "$AB=FB$" read $AD=FB$; and for "$\triangle ADK$" read $\triangle ADB$.

Page 47, first line, omit "Professor of Mathematics, Bowdoin College."

Page 54, thirteenth line of the solution, for "$m$" read $m^2$.

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## THE MOTION OF A PROJECTILE IN A MEDIUM RESISTING AS THE CUBE OF THE VELOCITY.


By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics, Defiance College, Defiance, Ohio.


For the motion of a projectile in a medium resisting as the *square* of the velocity, works on Mechanics, after assuming the fundamental equations of translatory motion,

$$\left|\begin{array}{l} \Sigma P\cos\alpha - \Sigma m(d^2x/dt^2) = 0 \ldots\ldots(\alpha), \\ \Sigma P\cos\beta - \Sigma m(d^2y/dt^2) = 0 \ldots\ldots(\beta), \\ \Sigma P\cos\gamma - \Sigma m(d^2z/dt^2) = 0 \ldots\ldots(\gamma), \end{array}\right| \ldots\ldots(1),$$

easily deduce the following equations:

$$\left|\begin{array}{l} M(d^2x/dt^2) = Mg\cos\alpha + Mcv^2\cos\alpha' \ldots\ldots(\alpha), \\ M(d^2y/dt^2) = Mg\cos\beta + Mcv^2\cos\beta' \ldots\ldots(\beta), \\ M(d^2z/dt^2) = Mg\cos\gamma + Mcv^2\cos\gamma' \ldots\ldots(\gamma), \end{array}\right| \ldots\ldots(2).$$

Let $w$ be the *terminal* velocity of the projectile in the medium, and $W$ the weight of the projectile in pounds; then the resistance of the air, when the velocity of the projectile is $v$, is equivalent to a force of $R=(v/w)^3 W$ pounds. At the same time, the gravitational retardation produced is equivalent to a force $G=(v/w)^3 g$ pounds.

Refer the motion of the projectile to oblique co-ordinate axes. Let the axis $OX$ extend in the direction of projection at the point of infinite velocity, and the axis $OY$ extend vertically downwards. In other words, cause the trajectory to lie in the vertical *co-ordinate-plane* $XOY$. This will enable us practically to *ignore* the co-ordinate axis $OZ$, and the equation ($\gamma$) in (1) and (2).

In the case under consideration, $\cos\alpha=0$, $\cos\beta=-1$, $\cos\gamma=0$, $\cos\alpha'=-(dx/ds)$, $\cos\beta'=-(dy/ds)$, $\cos\gamma'=(dz/ds)$, and $v=ds/dt$.

Transforming (2) under these conditions,

$$\left|\begin{array}{l} \dfrac{d^2x}{dt^2} = -\dfrac{g}{w^3}\left(\dfrac{ds}{dt}\right)^3\dfrac{dx}{ds}\ldots\ldots(\alpha), \\ \dfrac{d^2y}{dt^2} = -\dfrac{g}{w^3}\left(\dfrac{ds}{dt}\right)^3\dfrac{dy}{ds}+g\ldots\ldots(\beta), \\ \dfrac{d^2z}{dt^2} = -\dfrac{g}{w^3}\left(\dfrac{ds}{dt}\right)^3\dfrac{dz}{ds}\ldots\ldots(\gamma), \end{array}\right| \ldots\ldots(3).$$

From ($\beta$) and ($\alpha$) in (3),

$$\left(\frac{ds}{dt}\right)^3 = -\frac{w^3}{g}\left(\frac{d^2y}{dt^2}-g\right)\frac{ds}{dy} = -\frac{w^3}{g}\left(\frac{d^2x}{dt^2}\right)\frac{ds}{dx}\ldots\ldots(a).$$

Dividing ($a$) by $ds/dt$, etc.,

$$\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dy}-\left(\frac{d^2x}{dt^2}\right)\frac{dt}{dx} = g\frac{dt}{dy}\ldots\ldots(b).$$

Multiplying ($b$) by $(dx/dt)(dy/dt)$, etc.,

$$\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{d^2x}{dt^2}\cdot\frac{dy}{dt} = g\frac{dx}{dt}\ldots\ldots(c).$$

$$\therefore\left(\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{d^2x}{dt^2}\cdot\frac{dy}{dt}\right)\frac{dt}{dx}\Big/\left(\frac{dx}{dt}\right)^2 = g\frac{dx}{dt}\Big/\left(\frac{dx}{dt}\right)^2\ldots\ldots(d).$$

$$\therefore\frac{d}{dt}\left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)\frac{dt}{dx} = g\left(1\Big/\frac{dx}{dt}\right)\ldots\ldots(e).$$

$$\therefore\frac{dp}{dt} = g\frac{dt}{dx},\ \text{or}\ \frac{dp}{dt}\cdot\frac{dx}{dt} = g\ldots\ldots(f),$$